Time Value of Money and Investment Analysis

Explanations and Spreadsheet Applications for Agricultural and Agribusiness Firms

Part I.

by

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Part I.

TIME VALUE OF MONEY AND INVESTMENT ANALYSIS

INTRODUCTION

This document contains explanations and illustrations of common *Time Value of Money* problems facing agricultural and agribusiness firms. The accompanying spreadsheet files contain applications that mirror each section of the booklet, and provide the tools to do real-time computations and illustrations of the ideas presented in the text. Taken together, they provide the capacity to analyze and solve a wide array of real-world investment problems.

Time Value of Money problems refer to situations involving the exchange of something of value (money) at different points in time. In a basic sense, all investments involve the exchange of money at one point in time for the rights to the future cash flows associated with that investment. Expressing all of the values that are exchanged in terms of a common medium of exchange, or *money*, allows different sets of products or services to be compared in terms of a single standard of value (e.g., dollars). However, the passage of time between the outflows and inflows in a typical investment situation results in different *current* values associated with cash flows that occur at different points in time. Thus, it is not possible to assess an investment simply by adding up the total cash inflows and outflows and determining if they are positive or negative without first considering *when* the cash flows occur.

There are four primary reasons why a dollar to be received in the future is worth less than a dollar to be received immediately. The first and most obvious reason is the presence of positive rates of inflation which reduce the purchasing power of dollars through time. Secondly, a dollar today is worth more today than in the future because of the opportunity cost of lost earnings -- that is, it could have been invested and earned a return between today and a point in time in the future. Thirdly, all future values are in some sense only ***promises,*** and contain some uncertainty about their occurrence. As a result of the risk of default or nonperformance of an investment, a dollar in hand today is worth more than an expected dollar in the future. Finally, human preferences typically involve impatience, or the preference to consume goods and services now rather than in the future.

Interest rates represent the price paid to use money for some period of time. Interest rates are positive to compensate lenders (savers) for foregoing the use of money for some interval of time. The interest rate must offset the collective effects of the four reasons cited above for preferring a dollar today to a dollar in the future. The interest rate per period along with the other information about the sizes and timings of cash flows permit meaningful investment analyses to be conducted.

Unfortunately, typical investment decisions are much more complicated than simply calculating the expected cash flows and interest rates involved. Included in real-world analyses may be investment options with different length lives, different sized investments, different financing terms, differing tax implications, and the overall feasibility of making the initial investment. In response, the materials in this document and the accompanying spreadsheets were developed to assist in placing each of these issues into a context that permits meaningful comparisons across differing investment situations. In each case, the cash flows associated with an investment are converted to similar terms and then converted to their equivalent values at a common point in time using tools and techniques that collectively comprise the concepts known as the *Time Value of Money*.

The materials in this document are organized into three sections. The first section discusses the conceptual underpinnings of *time value of money* techniques along with the resulting mathematical expressions, and provides convenient summary of the formulas that are used to solve many time value of money problems. The second section discusses informational needs, alternative approaches to investment analysis, and common problems encountered in "real world" analyses of time value of money problems. The third section contains a collection of individual chapters devoted to descriptions of the spreadsheet applications for use in conducting meaningful investment analyses. In total, we hope this package is useful for learning and applying time value of money concepts to make better financial decisions. – ***F****arm* ***A****nalysis* ***S****olution* ***T****ools*

Time Value of Money and Investment Analysis

Part I: BASIC CONCEPTS AND TERMS

Time value of money problems arise in many different forms and situations. Thus, it is important to establish some common concepts and terminology to permit accurate characterization of their features. Among the most important characteristics of time value problems are: (i) the direction in time that cash flows are converted to equivalent values, (ii) whether there is a single cash flow, or a series of cash flows, and (iii) the decision variable or unknown value of the problem.

The first feature to establish involves the direction in time toward which cash flows are converted. *Compounding* refers to situations where a current value is being converted to its equivalent future value for comparison to another future value. *Discounting* involves moving back through time, or the conversion of a cash flow to be received in the future into its equivalent current value. The second important feature to establish is whether the cash flow type is a single payment at some point in time or a series of payments through time. Periodic payment, or series problems, can be solved as a collection of single-payment problems, but fortunately there are more convenient solution techniques for series problems than solving a set of single payment procedures. A further distinction of series problems that can be made is whether it involves a series with fixed payment size (commonly referred to as "uniform series" problems) or whether the series payment size is growing or declining through time. Finally, the decision variable or item whose value is being sought in the problem must be established. The collection of time value of money techniques can be used to solve for present values, future values, the payment size in a series, the interest rate or yield, or the length of time involved in a decision.

Although there are many variants of time value of money problems, they can nearly all be placed into one of six categories. The variants of each category of problem permit solutions for different decision variables, but each involves the same basic formulas. The following pages identify and describe the six basic categories of time value of money problems. Brief examples of each type of problem are provided to help develop identification skills for real world applications. There are also a series of example situations described in a companion document that can be used as a self-test for those interested in honing their skills at classifying and solving TVM problems.

Categories of Time Value of Money Problems

The six basic types of time value of money problems are described below. These six can also be described in terms of the elements introduced earlier to characterize problems: (i) the direction in time that cash flows are converted to equivalent values, (ii) whether the cash flow is a single value or a repeated series, and (iii) the decision variable or unknown value of the problem. Of the six basic types – the first three categories involve compounding, or the conversion of current and series payments to future values. Categories four through six involve discounting, or finding current values associated with future cash flows. Categories one and four apply to single payment problems and differ only by whether the future or present value is being sought. Categories two and five are used to address series payments rather than single payment situations and differ only by whether the future or present value is being sought. Categories three and six are employed when the size of the payment in a series is being sought when the total value of the series of payments is already known at some point in time. Thus, they differ from two and five respectively only by which item is the unknown or decision variable in the analysis. In each case, once the appropriate category is identified for the solution of a problem, the associated formula can be rearranged to solve for different variants of the problem. These six problem types are described more fully below with example situations in which they would each apply.

1. *Single-Payment Compound Amount (SPCA)*

This category refers to problems that involve a known single initial outlay invested at a specified interest rate and compounded at a regular basis. It is used when one needs to know the value to which the original single principal or investment will grow by the end of a specified time period. A savings deposit account that pays interest represents an SPCA problem when one desires to know how much an initial deposit will grow to by the end of a specific time period. Another example would be to find the value of a savings bond paying a known interest rate, at some point in time in the future. Variants of the formula used to solve this problem can be used to solve for (i) the length of time needed for an investment to double in value at a known interest rate, and (ii) the yield on an investment that doubled in valued over a known interval of time.

2. *Uniform Series Compound Amount (USCA)*

This category of problem involves known periodic payments invested at a regular intervals into an interest bearing account or interest paying investment that permits interest to be reinvested into the project. It is used to solve for the future value that this uniform series of payments of deposits grows into at compound interest, when continued for the specified length of time. This concept is complicated by the fact that each succeeding deposit earns interest for one less period than the preceding deposit. Examples of this application include solving for the size of a retirement account expected if regular monthly deposits are made into an interest paying investment account. Another example would be to solve for the value of a savings account for college expenses at the time a child turns 18, if annual deposits are made to their account. Life insurance policies' cash value computations utilize the formula associated with this problem as well. Variants of this problem include solving for: (i) at what point in time will an account be worth some amount if regular deposits of known size are made into the account with a known interest rate and regular compounding, and (ii) what rate of return is needed if known periodic payments are made into an account and a known future amount (e.g., enough to retire) is needed.

3. *Sinking Fund Deposit (SFD)*

A third variation of the compounding problem occurs when the desire is to make regular uniform deposits that will generate a *predetermined* amount by the end of a given period. The compound interest rate and the number of deposits to be made are known, but the size of the necessary deposits is unknown. For example, one might decide when a child is born to make monthly deposits toward a college education. If the goal is to have \$30,000 at the end of 18 years, for example, the formula associated with this category can be used to solve for the size of the required monthly deposits needed to meet that goal. Another example that often arises is to find the savings amount needed each period in an interest earning account to be able to purchase something of known value in the future. SFD calculations can also be used to find the size of the diversion in income needed to be able to retire a balloon payment on a loan when it comes due in the future. Variants of this problem include situations such as solving for the length of time one would need to work until retirement if regular deposits are made until retirement of a known size into an account paying a known interest rate, and if a known retirement account threshold must be reached. Or, the formula associated with the SFD problem can be used to solve for the required yield needed for a series of known investment contributions to grow to a given size in a specified interval of time.

4. *Single-Payment Present Value (SPPV)*

Single-payment present value problems involve calculations solving for the discounted value of a future single payment that results in an equivalent value in exchange today (present). Solving for the present value of a known future payment is the inverse of the problem of solving for the future value of a known present value. The only difference between SPCA and SPPV problems is the direction in time toward which money is being converted. In the SPPV problem, time takes on a negative value – that is the problem is used to transfer a future known value backwards in time to the present. Examples include calculations of the price to pay today for a pure discount bond, or the value today of a promise that someone makes to pay you a known amount at some point in time in the future. Variants of this problem, like SPCA problems, involve solving for the interest rate or time factors needed to convert a future value to its known equivalent present value under different circumstances.

5. *Uniform-Series Present Value (USPV)*

In this category, a series of payments of equal size is to be received at different points of time in the future, and the present value of the total series of payments is being sought. Although this type problem is conceptually equivalent to a series of SPPV problems, the formula involved is much simpler if the payments can be expressed as a series. For example, if one were entitled to receive fixed payments at the end of each year for five years, then there are really five SPPV problems with the sum of the results being equal to the USPV. Examples include calculation of the current value of a set of scheduled retirement payments, a series of sales receipts, or other situations in which there is a series of future cash inflows. Traditional investment theory asserts that "the value of an investment today is equal to the discounted sum of all future cash flows". That statement of equivalence between future cash flows and present value is the most general application of the formula associated with this category of time value of money problems. Variants of this problem include (i) calculation of "factoring" rates, of the implicit cost of borrowing if one were to sell a set of receivables for a known current amount, or, (ii) finding the length of time needed to retire an obligation if the periodic maximum payments and interest rate are known.

6. *Capital Recovery (CR)*

A problem closely related to the USPV is the capital recovery problem, also known as the loan amortization payment problem. In this case the present value is known, (the original loan balance which must be repaid) and the interest rate on unpaid remaining principal is known. In question is the size of equal payments (covering both interest and principal) which must be made each time period to exactly retire the entire remaining principal with the last payment. Typical lending situations provide the bulk of the examples of this problem with variants that are analogous to those in the USPV case. It should be noted that the difference between the USPV case and the CR case is whether the present value (e.g., initial loan amount) or size of payment is the unknown. Common variants in practice include (i) finding the maximum size loan that can be borrowed with a known income stream or debt repayment capacity, or (ii) finding the length of time over which a loan must be amortized for the loan payments to be of a given acceptable size.

Conceptualization and Solution of Time Value of Money Problems

The two necessary phases of solving time value of money problems are: 1) correctly identifying which type of problem exists and what factor is unknown; and 2) correctly applying the appropriate mathematical calculations to find the answer. Classification of problems is a skill that can be developed by carefully reducing the problem into its known and unknown values, and then ruling out approaches that do not apply. It is extremely helpful to draw a timeline associated with the cashflows of the problem to assist in problem classification and description. Once classified, the mathematical formulas contained below and in the spreadsheets supplied with this text can be used to find the actual values. These values are often necessary in making management or investment decisions.

It is useful to first provide standardized notation that can be used in solving each of the problems. A summary of the notation used is provided below.

<u>**Notation Summary:**</u>

P_t = Payment of size P at time t. Payments may differ through time.

V_o = present value, or sometimes PV, value at time 0.

V_t = general notation for total value at time t.

n = a period of time (could be month, half-year, year, etc) also a point in time, with the final period in time often given as N.

t = time index, especially common in continuously compounded problems, final point in time is often given as T.

A = annuity, or simply the periodic payment amount (always a constant amount)

m = number of interest rate compoundings per period of time.

r = interest rate per period of time.

exp, e, or e = base of the natural logarithm.

<u>Other conventions commonly employed in this bookelt and in other TVM materials:</u>

- The current time period, or present, is always time 0.
- Discrete time problems usually use "*n*" for intervals of time, and by convention, the payments flows occur at the *end* of the time interval unless otherwise indicated. Thus, a payment P_1 is a payment that occurs at the end of the first period.
- Continuous time problems usually use "*t*" to represent a point in time.
- Loan amounts are special cases of V_o and are sometimes written as L
- Bonds and investments paying fixed coupons represent special cases of P_t and are sometimes written as C_t to represent "coupon payment"

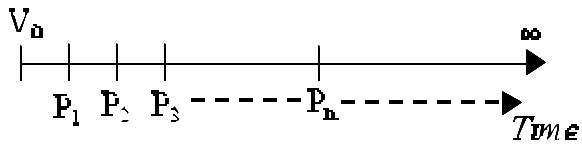
Derivation of formulas used for each category of problem.

This section provides the mathematical relationships and algorithms that are associated with each of the six categories above. In addition, it contains the continuous-time formulas that are often used under continuous compounding, or as simplifications of the discrete time versions. All six of the formulas can be derived from the same basic principles, and therefore the relationships among the formulas should be apparent after working through this section. At the end of this section, each of the formulas is restated in summary form on a single page intended as one page pullout reference, and thus, this section can be skipped without loss of continuity or applicability.

To begin, consider an initial principal deposit, V_0, deposited in an interest bearing account that pays (compounds) a fixed interest rate, r, annually. After one period, the interest earnings of $P_1 = r*V_0$ could be withdrawn leaving the original balance in place to earn interest for the next period. After the second period, the same interest-only payment could be made from this account. In fact, the interest could be withdrawn each period in perpetuity and never disturb the original principal balance in the account. Thus, in exchange for the initial principal, the investor can receive a perpetual series of payments equal to $r*V_0$ which are all also equal to P_1 since all interest withdrawals would be equal. The relationship $r*V_0 - P_1$ is known as the fundamental capitalization formula and is usually rearranged and written as:

[1] $V_0 = \dfrac{P_1}{r}$,

This relationship can be depicted graphically as shown below:

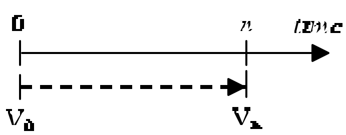


Next, consider the same deposit, but now assume the interest earnings are never withdrawn, but instead are left in the account to accumulate interest as well in all future periods. After one period, the account will be worth the initial principal plus interest earnings or $V_0 + r*V_0$. This equation can be rewritten as: $V_0(1+r)^1$. That amount, if left undisturbed for the second period will be worth its initial value at the beginning of the period or $V_0(1+r)^1$ plus $r*V_0(1+r)^1$ in interest earnings. This amount can be rewritten as $V_0(1+r)(1+r)$ or equivalently $V_0(1+r)^2$. Each successive period will result in end of that period's value equal to its beginning period value times $(1+r)$. Thus, after n periods, an initial deposit of V_0 will grow to:

[2] $V_0(1+r)^n = V_n$

which can be graphically depicted as:

SPCA



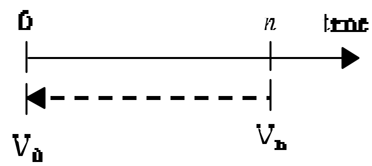
The value $(1+r)^n$ is the SPCA interest rate factor which, when multiplied by any size initial deposit gives its future value after n periods at interest rate per period of r, compounded once per period.

Once the present value and future value are linked through the interest rate and time relationship, equation [2] can be rearranged to solve for V_0 in terms of V_n giving:

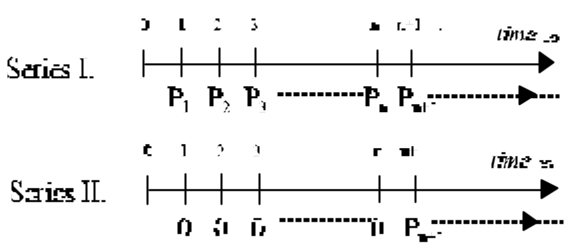
[3] $$V_0 = \frac{V_n}{(1+r)^n} ,$$

which is often rewritten as $V_0 = V_n(1+r)^{-n}$. The negative exponent in this version highlights the idea of moving backward in time to get back to a present value. The relationship can be graphically depicted as:

SPPV



Combining these relationships permits the valuation of series of payments, although the algebra is a bit more involved. To begin, consider the perpetual series of payments of size P_1 beginning at the end of the first period and lasting forever (labeled "Series I." in the figure below). From the fundamental capitalization formula, it is known that its current value is simply P_1/r. Now consider a second series of payments (labeled "Series II." in the figure below), but this time the first payment is received at time period $n+1$, and at the end of every period thereafter. At the future point in time n, that series is worth P_{n+1}/r. If you begin with series I and subtract series II, what remains is a series of payments arriving at the end of each of the first n periods into the future and then zero thereafter -- or a uniform series lasting n periods. Graphically,



Using the formula for SPPV, series II has a current value of $(P_{n+1}/r)(1+r)^{-n}$. Since the payments are of equal size at all points in time, the subscripts can all be written as 1 and each payment referred to as P_1.

Thus, the formula for a uniform series present value can be written as V_0 = (Series I present value) minus (Series II present value) or:

[4] $$V_0 = \frac{P_1}{r} - \left(\frac{P_1}{r}\right)\left(\frac{1}{1+r}\right)^n$$

Factoring out P, multiplying through both parts on the right hand side by $(1+r)^n$ results in the typical expression for the USPV relationship:

[5] $$V_0 = P_1\left[\frac{(1+r)^n - 1}{r(1+r)^n}\right].$$

Equation [5] can be rearranged to find the size of payment per period for n periods at interest rate r that has present value equal to V_0. The result is the capital recovery formula (CR), or as more commonly called, the loan amortization formula. Rearranging eq. [5] for P_1 gives:

[6] $$P_1 = V_0\left[\frac{r(1+r)^n}{(1+r)^n - 1}\right],$$

which is used to find the size of equal periodic payment at the end of each period for n periods at interest rate r that will retire an initial loan principal of size V_0.

The uniform series compound amount (USCA) differs from the USPV formula by solving for the future value rather than the present value of a series of payments of known size for a known number of periods and a known interest rate. Given the SPCA formula that links present to future values, the USCA formula can be found from the USPV by simply compounding the present value from the USPV equation to the end of the time horizon. Algebraically, multiply both sides of eq. [4] by $(1+r)^n$ to get:

[7] $$V_0(1+r)^n = P_1\left[\frac{(1+r)^n - 1}{r(1+r)^n}\right](1+r)^n,$$

and recognizing that $V_0(1+r)^n = V_n$, and then rearranging the expression results in the formula for the USCA:

[8] $$V_n = P_1\left[\frac{(1+r)^n - 1}{r}\right].$$

The sinking fund deposit formula (SFD) can easily be found from the above formula [8] by rearranging for the size of periodic payment, P_1 that results in a future value of V_n after n periodic payments at a known interest rate. Doing so results in the following:

[9] $$P_1 = V_n\left[\frac{r}{(1+r)^n - 1}\right]$$

Impact of Compounding Frequency

Interest rates are typically stated in annual form. However, many times the payment frequency or compounding interval differs from the annual rate. For example, a savings account may have an associated annual interest rate of 8% but have interest earnings that are credited to the account on a quarterly basis. The result is that part of the interest is available earlier and itself earns interest over the remainder of the investment period. Whenever the frequency of compounding and the interest rate time interval differ, adjustments must be made to the formulas above to account for the more or less frequent compounding. Fortunately, the adjustment is very simple involving only the interest rate per period, or r; the length of time, or n; and the frequency of compounding per period, or m. In each case, the interest rate is divided by m and the number of time periods is multiplied by m. When $m = 1$, it can be omitted from the equations above. When it differs from 1, it has the effect of converting the interest rate per year (or other interval) to an interest rate per compounding period and the effect of converting the number of years (or other interval) to the number of total payment periods. Some examples will help illustrate. Suppose you deposited \$100 into an account paying 8% interest per year and left it undisturbed for 10 years. Using the standard SPCA formula, the terminal value could be found as $Vn = V_0(1+r)^n$ or in this case $100(1.08)^{10}$ = \$215.89. If instead, the interest earned were credited quarterly, then m=4 and the resulting formula is: $Vn = V_0(1+r/m)^{n*m}$ or in this case $100(1.02)^{40}$ = \$220.80. The increased value with more frequent compounding simply represents the additional interest earned through time on the interest payments that were credited to the account earlier. If the interest were credited monthly in this case, the final value would have been \$221.96. Under daily compounding, the resulting calculation is $\$100(1+.08/365)^{365*10}$ = \$222.53. *Naturally*, more frequent compounding results in higher values in future value calculations. And, it is *natural* to ask what the *limiting* value that frequency of compounding implies about the future value. And so, with no further plays on words, the natural logarithm is introduced with its implications to the time value of money.

Note that $V_n = V_0(1+r/m)^{n*m}$ can be rewritten as:

[10] $$V_n = V_0\left(1+\frac{1}{m/r}\right)^{\frac{m}{r}*(r*n)}$$

and recalling that $\lim_{h\to\infty}\left(1+\frac{1}{h}\right)^h = e$, where e is the base of the natural logarithm

eq.[10] can be evaluated at the limit of the frequency of compounding as m approaches infinity or:

[11] $$\lim_{m\to\infty}\left[V_0\left\{\left(1+\frac{1}{m/r}\right)^{\frac{m}{r}}\right\}^{r*n}\right] = V_0e^{r*n},$$

because the term in the curly braces has a limiting value of *e*, or the base of the natural logarithm. Hence, under continuous compounding, the $100 deposit at 8% annual interest for 10 years would be worth $\$100e^{.08*10} = \$222.55$.

This formula can be easily rearranged for the present value that a future known value represents. In this case, the final expression in eq. [11] can be rewritten as:

[12] $V_0 = V_n e^{-r*n}$.

Summary of Materials in Part I

The above materials are meant to help the user to (i) understand reasons to use time value of money approaches in problems involving cash flows through time, (ii) understand the categories of problems that require application of time value of money concepts, (iii) be able to classify problems into the appropriate category for solution, (iv) understand the background and derivation of the formulas used to solve TVM problems. To provide a simple and compact reference, the major formulas are repeated below in summary form with descriptions identifying their application purposes. Each of the formulas described on the following page is included in the accompanying spreadsheet with utilities to make their use simple and direct, thus avoiding many of the calculation errors that can occur when working through the formulas by hand with a calculator.

Summary of TVM Formulas

1. **(SPCA) Single payment compounded future amount.** *(Unknown value is future amount, known values are: interest rate per period, initial principal and the number of periods)*

$$V_n = V_0(1+r)^n$$

And under continuous compounding,

$$V_n = V_0 e^{rt}$$

2. **(USCA) Uniform series compound amount.** *(Unknown value is future amount, known values are: interest rate per period, periodic payments, and the number of periods).*

$$V_n = P_1\left[\frac{(1+r)^n - 1}{r}\right]$$

3. **(SFD) Sinking fund deposit.** *(Unknown value is size of periodic payment, known* values are: interest rate per period, future value, and the number of periods).

$$P_1 = V_n\left[\frac{r}{(1+r)^n - 1}\right]$$

4. **(SPPV) Single payment present value.** *(Unknown value is present value, known values are: interest rate per period, future value, and the number of periods*).

$$V_0 = V_n(1+r)^{-n}$$

And under continuous discounting

$$V_0 = V_n e^{-rt}$$

5. **(USPV) Uniform series present value**. *(Unknown value is present value, known values are: interest rate per period, periodic payments, and the number of periods).*

$$V_0 = P_1\left[\frac{(1+r)^n - 1}{r(1+r)^n}\right]$$

6. **(CR) Capital recovery or the loan payment problem**. *(Unknown value is the size of payments, known values are: interest rate per period, initial principal value, and the number of periods).*

$$P_1 = V_0\left[\frac{r(1+r)^n}{(1+r)^n - 1}\right]$$

7. Fundamental capitalization formula: *(Unknown value is present value, known values are: interest rate per period, periodic perpetual payments).*

$$V_o = P_1/r$$